Lecture 12

More on Transmission Lines

12.1 Terminated Transmission Lines

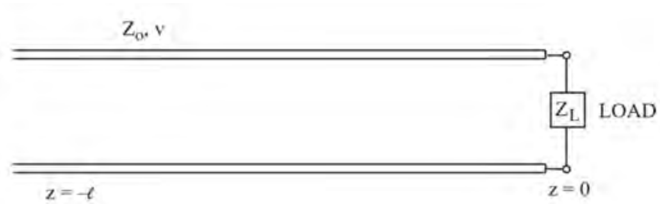

Figure 12.1: A schematic for a transmission line terminated with an impedance load Z_L at $z = 0$.

As mentioned before, transmission line theory is indispensable in electromagnetic engineering. It is similar to one-dimensional form of Maxwell's equations, and can be thought of as Maxwell's equations in its simplest form. Therefore, it entails a subset of the physics seen in the full Maxwell's equations.

For an infinitely long transmission line, the solution consists of the linear superposition of a wave traveling to the right plus a wave traveling to the left. If transmission line is terminated by a load as shown in Figure 12.1, a right-traveling wave will be reflected by the load, and in general, the wave on the transmission line will be a linear superposition of the left and right traveling waves. We will assume that the line is lossy first and specialize it to the lossless case later. Thus,

$$V(z) = a_+ e^{-\gamma z} + a_- e^{\gamma z} = V_+(z) + V_-(z) \tag{12.1.1}$$

This is a linear system; hence, we can define the right-going wave $V_+(z)$ to be the input, and that the left-going wave $V_-(z)$ to be the output as due to the reflection of the right-going wave $V_+(z)$. Or we can define the amplitude of the left-going reflected wave a_- to be linearly related to the amplitude of the right-going or incident wave a_+. In other words, at $z=0$, we can let

$$V_-(z=0) = \Gamma_L V_+(z=0) \tag{12.1.2}$$

thus, using the definition of $V_+(z)$ and $V_-(z)$ as implied in (12.1.1), we have

$$a_- = \Gamma_L a_+ \tag{12.1.3}$$

where Γ_L is the termed the reflection coefficient. Hence, (12.1.1) becomes

$$V(z) = a_+ e^{-\gamma z} + \Gamma_L a_+ e^{\gamma z} = a_+ \left(e^{-\gamma z} + \Gamma_L e^{\gamma z}\right) \tag{12.1.4}$$

The corresponding current $I(z)$ on the transmission line is given by using the telegrapher's equations as previously defined. By recalling that

$$\frac{\mathrm{d}V}{\mathrm{d}z} = -j\omega L I$$

then for the general case,

$$I(z) = -\frac{1}{Z}\frac{\mathrm{d}V}{\mathrm{d}z} = \frac{a_+}{Z}\gamma(e^{-\gamma z} - \Gamma_L e^{\gamma z}) \tag{12.1.5}$$

where $\gamma = \sqrt{ZY} = \sqrt{(j\omega L + R)(j\omega C + G)}$, and

$$Z = j\omega L + R, \qquad Y = j\omega C + G$$

In the lossless case when $R = G = 0$, $\gamma = j\beta$ where $\beta = \sqrt{LC}$ as a sanity check. But in general, $Z/\gamma = \sqrt{Z/Y} = Z_0$, the characteristic impedance of the transmission line. Thus, from (12.1.5),

$$I(z) = \frac{a_+}{Z_0}\left(e^{-\gamma z} - \Gamma_L e^{\gamma z}\right) \tag{12.1.6}$$

Notice the sign change in the second term of the above expression.

Similar to Γ_L, a general reflection coefficient (which is a function of z) relating the left-traveling and right-traveling wave at location z can be defined such that

$$\Gamma(z) = \frac{V_-(z) = a_- e^{\gamma z}}{V_+(z) = a_+ e^{-\gamma z}} = \frac{a_- e^{\gamma z}}{a_+ e^{-\gamma z}} = \Gamma_L e^{2\gamma z} \tag{12.1.7}$$

Of course, $\Gamma(z=0) = \Gamma_L$. Furthermore, due to the V-I relation at an impedance load, we must have[1]

$$\frac{V(z=0)}{I(z=0)} = Z_L \tag{12.1.8}$$

[1]One can also look at this from a differential equation viewpoint that this is a boundary condition.

or that using (12.1.4) and (12.1.6) with $z = 0$, the left-hand side of the above can be rewritten, and we have

$$\frac{1+\Gamma_L}{1-\Gamma_L} Z_0 = Z_L \tag{12.1.9}$$

From the above, we can solve for Γ_L in terms of Z_L/Z_0 to get

$$\Gamma_L = \frac{Z_L/Z_0 - 1}{Z_L/Z_0 + 1} = \frac{Z_L - Z_0}{Z_L + Z_0} \tag{12.1.10}$$

Thus, given the termination load Z_L and the characteristic impednace Z_0, the reflection coefficient Γ_L can be found, or vice versa. Or given Γ_L, the normalized load impedance, Z_L/Z_0, can be found. It is seen that $\Gamma_L = 0$ if $Z_L = Z_0$. Thus a right-traveling wave will not be reflected and the left-traveling is absent. This is the case of a matched load. When there is no reflection, all energy of the right-traveling wave will be totally absorbed by the load.

In general, we can define a generalized impedance at $z \neq 0$ to be

$$\begin{aligned} Z(z) = \frac{V(z)}{I(z)} &= \frac{a_+(e^{-\gamma z} + \Gamma_L e^{\gamma z})}{\frac{1}{Z_0} a_+(e^{-\gamma z} - \Gamma_L e^{\gamma z})} \\ &= Z_0 \frac{1+\Gamma_L e^{2\gamma z}}{1-\Gamma_L e^{2\gamma z}} = Z_0 \frac{1+\Gamma(z)}{1-\Gamma(z)} \end{aligned} \tag{12.1.11}$$

or

$$Z_n(z) = Z(z)/Z_0 = \frac{1+\Gamma(z)}{1-\Gamma(z)} \tag{12.1.12}$$

where $Z_n(z)$ is the normalized generalized impedance, and $\Gamma(z)$ is as defined in (12.1.7). Conversely, one can write the above as

$$\Gamma(z) = \frac{Z(z)/Z_0 - 1}{Z(z)/Z_0 + 1} = \frac{Z(z) - Z_0}{Z(z) + Z_0} \tag{12.1.13}$$

Usually, a transmission line is lossless or has very low loss, and for most practical purpose, $\gamma = j\beta$. In this case, (12.1.11) becomes

$$Z(z) = Z_0 \frac{1+\Gamma_L e^{2j\beta z}}{1-\Gamma_L e^{2j\beta z}} \tag{12.1.14}$$

From the above, one can show that by setting $z = -l$, using (12.1.10), and after some algebra,

$$Z(-l) = Z_0 \frac{Z_L + jZ_0 \tan\beta l}{Z_0 + jZ_L \tan\beta l} \tag{12.1.15}$$

12.1.1 Shorted Terminations

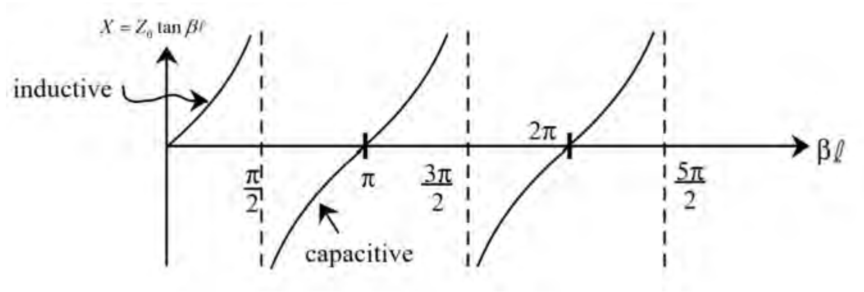

Figure 12.2: The input reactance (X) of a shorted transmission line as a function of its length l.

From (12.1.15) above, when we have a short such that $Z_L = 0$, then

$$Z(-l) = jZ_0 \tan(\beta l) = jX \tag{12.1.16}$$

Hence, the impedance remains reactive (pure imaginary) for all l, and can swing over all positive and negative imaginary values. One way to understand this is that when the transmission line is shorted, the right and left traveling wave set up a standing wave with nodes and anti-nodes. At the nodes, the voltage is zero while the current is maximum. At the anti-nodes, the current is zero while the voltage is maximum. Hence, a node resembles a short while an anti-node resembles an open circuit. Therefore, at $z = -l$, different reactive values can be observed as shown in Figure 12.2.

When $\beta \ll l$, then $\tan \beta l \approx \beta l$, and (12.1.16) becomes

$$Z(-l) \cong jZ_0 \beta l \tag{12.1.17}$$

After using that $Z_0 = \sqrt{L/C}$ and that $\beta = \omega\sqrt{LC}$, (12.1.17) becomes

$$Z(-l) \cong j\omega L l \tag{12.1.18}$$

The above implies that a short length of transmission line connected to a short as a load looks like an inductor with $L_{\text{eff}} = Ll$, since much current will pass through this short producing a strong magnetic field with stored magnetic energy. Remember here that L is the line inductance, or inductance per unit length.

12.1.2 Open terminations

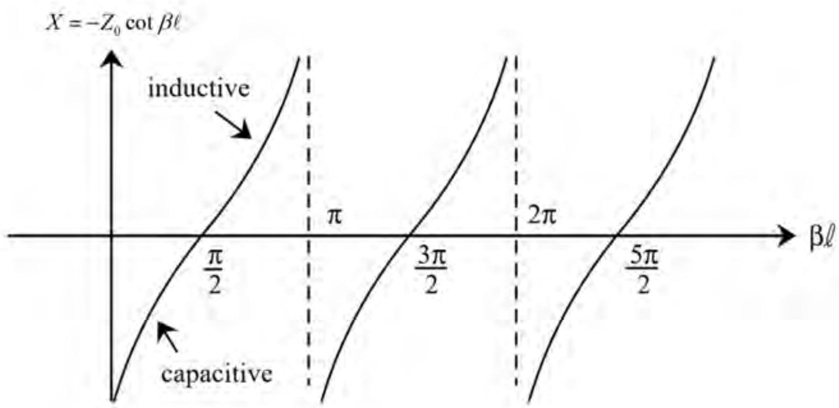

Figure 12.3: The input reactance (X) of an open transmission line as a function of its length l.

When we have an open circuit such that $Z_L = \infty$, then from (12.1.15) above

$$Z(-l) = -jZ_0 \cot(\beta l) = jX \tag{12.1.19}$$

Again, as shown in Figure 12.3, the impedance at $z = -l$ is purely reactive, and goes through positive and negative values due to the standing wave set up on the transmission line.

Then, when $\beta l \ll l$, $\cot(\beta l) \approx 1/\beta l$

$$Z(-l) \approx -j\frac{Z_0}{\beta l} \tag{12.1.20}$$

And then, again using $\beta = \omega\sqrt{LC}$, $Z_0 = \sqrt{L/C}$

$$Z(-l) \approx \frac{1}{j\omega Cl} \tag{12.1.21}$$

Hence, an open-circuited terminated short length of transmission line appears like an effective capacitor with $C_{\text{eff}} = Cl$. Again, remember here that C is line capacitance or capacitance per unit length.

But the changing length of l, one can make a shorted or an open terminated line look like an inductor or a capacitor depending on its length l. This effect is shown in Figures 12.2 and 12.3. Moreover, the reactance X becomes infinite or zero with the proper choice of the length l. These are resonances or anti-resonances of the transmission line, very much like an LC tank circuit. An LC circuit can look like an open or a short circuit at resonances and depending on if they are connected in parallel or in series.

12.2 Smith Chart

In general, from (12.1.14) and (12.1.15), a length of transmission line can transform a load Z_L to a range of possible complex values $Z(-l)$. To understand this range of values better, we can use the Smith chart (invented by P.H. Smith 1939 before the advent of the computer) [81]. The Smith chart is essentially a graphical calculator for solving transmission line problems. Equation (12.1.13) indicates that there is a unique map between the normalized impedance $Z_n(z) = Z(z)/Z_0$ and reflection coefficient $\Gamma(z)$. In the normalized impedance form where $Z_n = Z/Z_0$, from (12.1.11) and (12.1.13)

$$\Gamma = \frac{Z_n - 1}{Z_n + 1}, \qquad Z_n = \frac{1 + \Gamma}{1 - \Gamma} \tag{12.2.1}$$

Equations in (12.2.1) are related to a bilinear transform in complex variables [82]. It is a conformal map that maps circles to circles. Such a map is shown in Figure 12.4, where lines on the right-half of the complex Z_n plane are mapped to the circles on the complex Γ plane. Since straight lines on the complex Z_n plane are circles with infinite radii, they are mapped to circles on the complex Γ plane. The Smith chart allows one to obtain the corresponding Γ given Z_n and vice versa as indicated in (12.2.1), but using a graphical calculator or the Smith chart.

Notice that the imaginary axis on the complex Z_n plane maps to the circle of unit radius on the complex Γ plane. All points on the right-half plane are mapped to within the unit circle. The reason being that the right-half plane of the complex Z_n plane corresponds to passive impedances that will absorb energy. Hence, such an impedance load will have reflection coefficient with amplitude less than one, which are points within the unit circle.

On the other hand, the left-half of the complex Z_n plane corresponds to impedances with negative resistances. These will be active elements that can generate energy, and hence, yielding $|\Gamma| > 1$, and will be outside the unit circle on the complex Γ plane.

Another point to note is that points at infinity on the complex Z_n plane map to the point at $\Gamma = 1$ on the complex Γ plane, while the point zero on the complex Z_n plane maps to $\Gamma = -1$ on the complex Γ plane. These are the reflection coefficients of an open-circuit load and a short-circuit load, respectively. For a matched load, $Z_n = 1$, and it maps to the zero point on the complex Γ plane implying no reflection.

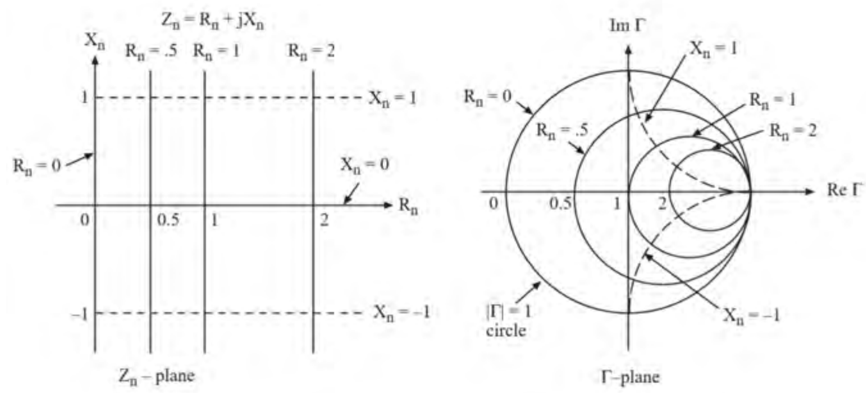

Figure 12.4: Bilinear map of the formulae $\Gamma = \frac{Z_n-1}{Z_n+1}$, and $Z_n = \frac{1+\Gamma}{1-\Gamma}$. The chart on the right, called the Smith chart, allows the values of Z_n to be determined quickly given Γ, and vice versa.

The Smith chart also allows one to quickly evaluate the expression

$$\Gamma(-l) = \Gamma_L e^{-2j\beta l} \tag{12.2.2}$$

and its corresponding Z_n not by using (12.2.1) via a calculator, but by using a graphical calculator. Since $\beta = 2\pi/\lambda$, it is more convenient to write $\beta l = 2\pi l/\lambda$, and measure the length of the transmission line in terms of wavelength. To this end, the above becomes

$$\Gamma(-l) = \Gamma_L e^{-4j\pi l/\lambda} \tag{12.2.3}$$

For increasing l, one moves away from the load to the generator (or source), l increases, and the phase is decreasing because of the negative sign. So given a point for Γ_L on the Smith chart, one has negative phase or decreasing phase by rotating the point clockwise. Also, due to the $\exp(-4j\pi l/\lambda)$ dependence of the phase, when $l = \lambda/4$, the reflection coefficient rotates a half circle around the chart. And when $l = \lambda/2$, the reflection coefficient will rotate a full circle, or back to the original point.

Also, for two points diametrically opposite to each other on the Smith chart, Γ changes sign, and it can be shown easily that the normalized impedances are reciprocal of each other. Hence, the Smith chart can also be used to find the reciprocal of a complex number quickly. A full blown Smith chart is shown in Figure 12.5.

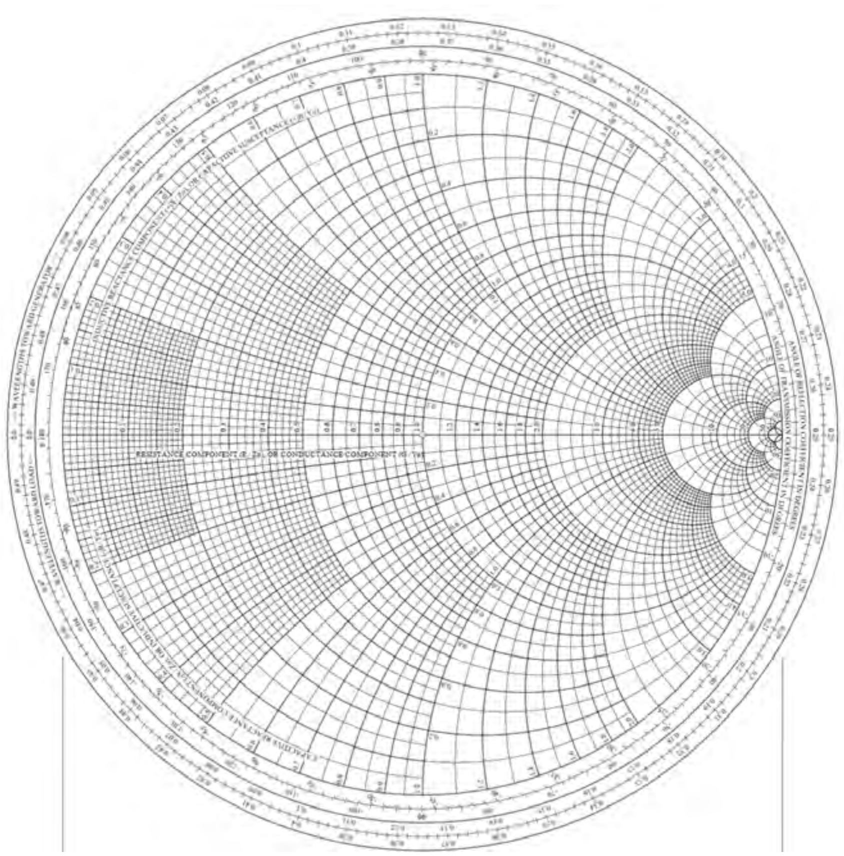

Figure 12.5: The Smith chart in its full glory. It was invented in 1939 before the age of digital computers, but it still allows engineers to do mental estimations and rough calculations with it, because of its simplicity.

12.3 VSWR (Voltage Standing Wave Ratio)

Due to the interference between that forward traveling wave and the backward traveling wave, $V(z)$ is a function of position z on a terminated transmission line and it is given as

$$\begin{aligned} V(z) &= V_0 e^{-j\beta z} + V_0 e^{j\beta z}\Gamma_L \\ &= V_0 e^{-j\beta z}\left(1+\Gamma_L e^{2j\beta z}\right) \\ &= V_0 e^{-j\beta z}\left(1+\Gamma(z)\right) \end{aligned} \tag{12.3.1}$$

where we have used (12.1.7) for $\Gamma(z)$ with $\gamma = j\beta$. Hence, $V(z)$ is not a constant but dependent on z, or

$$|V(z)| = |V_0||1 + \Gamma(z)| \tag{12.3.2}$$

For lack of a better name, this is called the standing wave, even though it is not truly a standing wave.

In Figure 12.6, the relationship between variation of $1 + \Gamma(z)$ as z varies is shown.

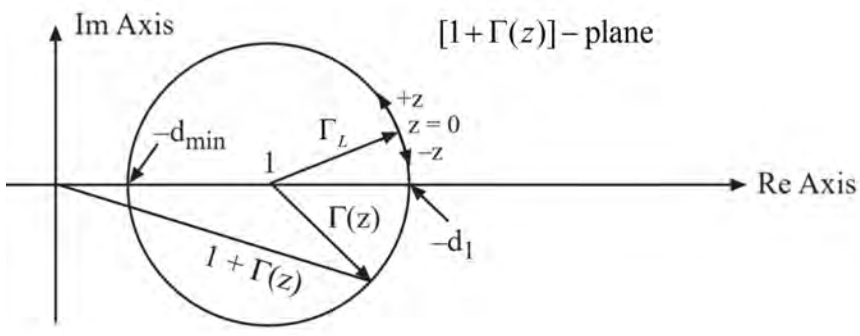

Figure 12.6: The voltage amplitude on a transmission line depends on $|V(z)|$, which is proportional to $|1 + \Gamma(z)|$ per equation (12.3.2). This figure shows how $|1 + \Gamma(z)|$ varies as z varies on a transmission line.

Using the triangular inequality, one gets

$$|V_0|(1 - |\Gamma(z)|) \leq |V(z)| \leq |V_0|(1 + |\Gamma(z)|) \tag{12.3.3}$$

But from (12.1.7) and that $\gamma = j\beta$, $|\Gamma(z)| = |\Gamma_L|$; hence

$$V_{\min} = |V_0|(1 - |\Gamma_L|) \leq |V(z)| \leq |V_0|(1 + |\Gamma_L|) = V_{\max} \tag{12.3.4}$$

The voltage standing wave ratio, VSWR is defined to be

$$\text{VSWR} = \frac{V_{\max}}{V_{\min}} = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|} \tag{12.3.5}$$

Conversely, one can invert the above to get

$$|\Gamma_L| = \frac{\text{VSWR} - 1}{\text{VSWR} + 1} \tag{12.3.6}$$

Hence, the knowledge of voltage standing wave pattern (VSWP), as shown in Figure 12.7, yields the knowledge of $|\Gamma_L|$ the amplitude of Γ_L. Notice that the relations between VSWR and $|\Gamma_L|$ are homomorphic to those between Z_n and Γ. Therefore, the Smith chart can also be used to evaluate the above equations.

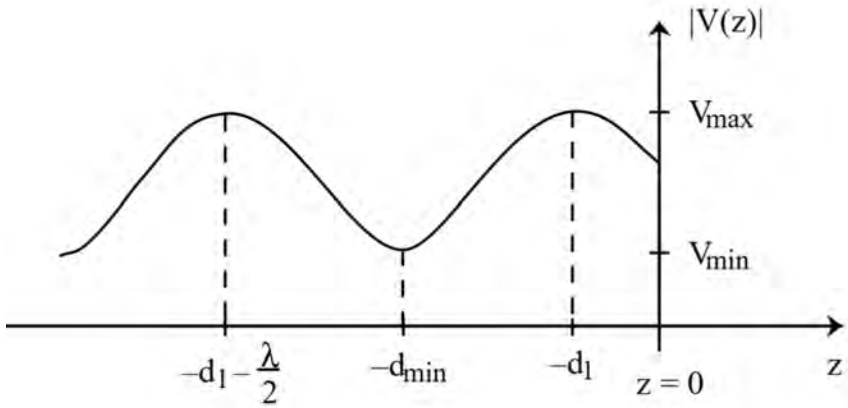

Figure 12.7: The voltage standing wave pattern (VSWP) as a function of z on a load-terminated transmission line.

The phase of Γ_L can also be determined from the measurement of the voltage standing wave pattern. The location of Γ_L in Figure 12.6 is determined by the phase of Γ_L. Hence, the value of d_1 in Figure 12.6 is determined by the phase of Γ_L as well. The length of the transmission line waveguide needed to null the original phase of Γ_L to bring the voltage standing wave pattern to a maximum value at $z = -d_1$ is shown in Figure 12.7. Hence, d_1 is the value where the following equation is satisfied:

$$|\Gamma_L| e^{j\phi_L} e^{-4\pi j(d_1/\lambda)} = |\Gamma_L| \tag{12.3.7}$$

Thus, by measuring the voltage standing wave pattern, one deduces both the amplitude and phase of Γ_L. From the complex value Γ_L, one can determine Z_L, the load impedance.

From the above, one surmises that measuring the impedance of a device at microwave frequency is a tricky business. At low frequency, one can use an ohm meter with two wire probes to do such a measurement. But at microwave frequency, two pieces of wire become inductors, and two pieces of metal become capacitors. More sophisticated ways to measure the impedance need to be designed as described above.

In the old days, the voltage standing wave pattern was measured by a slotted-line equipment which consists of a coaxial waveguide with a slot opening as shown in Figure 12.8. A field probe can be put into the slotted line to determine the strength of the electric field inside the coax waveguide.

Figure 12.8: A slotted-line equipment which consists of a coaxial waveguide with a slot opening at the top to allow the measurement of the field strength and hence, the voltage standing wave pattern in the waveguide (courtesy of Microwave101.com).

A typical experimental setup for a slotted line measurement is shown in Figure 12.9. A generator source, with low frequency modulation, feeds microwave energy into the coaxial waveguide. The isolator, allowing only the unidirectional propagation of microwave energy, protects the generator. The attenuator protects the slotted line equipment. The wavemeter is an adjustable resonant cavity. When the wavemeter is tuned to the frequency of the microwave, it siphons off some energy from the source, giving rise to a dip in the signal of the SWR meter (a short for voltage-standing-wave-ratio meter). Hence, the wavemeter measures the frequency of the microwave.

The slotted line probe is usually connected to a square law detector with a rectifier that converts the microwave signal to a low-frequency signal. In this manner, the amplitude of the voltage in the slotted line can be measured with some low-frequency equipment, such as the SWR meter. Low-frequency equipment is a lot cheaper to make and maintain. That is also the reason why the source is modulated with a low-frequency signal. At low frequencies, circuit theory prevails, engineering and design are a lot simpler.

The above describes how the impedance of the device-under-test (DUT) can be measured at microwave frequencies. Nowadays, automated network analyzers make these measurements a lot simpler in a microwave laboratory. More resource on microwave measurements can be found on the web, such as in [83].

Notice that the above is based on the interference of the two traveling wave on a terminated transmission line. Such interference experiments are increasingly difficult in optical frequencies because of the much shorter wavelengths. Hence, many experiments are easier to perform at microwave frequencies rather than at optical frequencies.

Many technologies are first developed at microwave frequency, and later developed at optical frequency. Examples are phase imaging, optical coherence tomography, and beam steering with phase array sources. Another example is that quantum information and quantum computing can be done at optical frequency, but the recent trend is to use artificial atoms working at microwave frequencies. Engineering with longer wavelength and larger component is easier; and hence, microwave engineering.

Another new frontier in the electromagnetic spectrum is in the terahertz range. Due to the dearth of sources in the terahertz range, and the added difficulty in having to engineer smaller components, this is an exciting and a largely untapped frontier in electromagnetic technology.

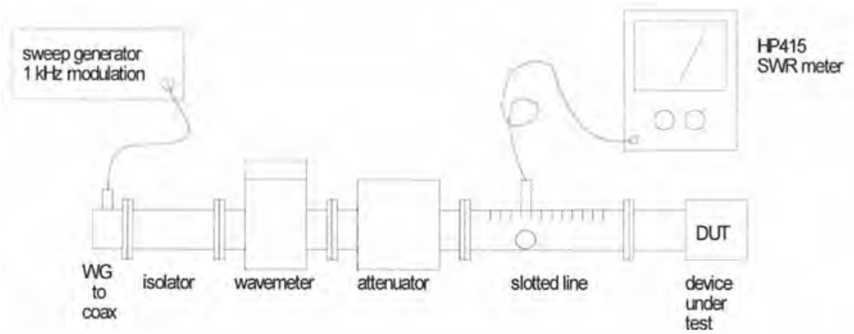

Figure 12.9: An experimental setup for a slotted line measurement (courtesy of Pozar and Knapp, U. Mass [84]).